I would tell this guy, “The first thing you must learn is the mathematics. And that involves, first, calculus. And in calculus, differentiation.”

Now, mathematics is a beautiful subject, and has its ins and outs, too, but we’re trying to figure out what the minimum amount we have to learn for *physics purposes* are. So the attitude that’s taken here is a “disrespectful” one towards the mathematics, for sheer efficiency only; I’m not trying to undo mathematics.

What we have to do is to learn to differentiate like we know how much is 3 and 5, or how much is 5 times 7, because that kind of work is involved so often that it’s good not to be confounded by it. When you write something down, you should be able to immediately differentiate it without even thinking about it, and without making any mistakes. You’ll find you need to do this operation all the time—not only in physics, but in all the sciences. Therefore differentiation is like the arithmetic you had to learn before you could learn algebra.

Incidentally, the same goes for algebra: there’s a lot of algebra. We are assuming that you can do algebra in your sleep, upside down, without making a mistake. We know it isn’t true, so you should also practice algebra: write yourself a lot of expressions, practice them, and don’t make any errors.

Errors in algebra, differentiation, and integration are only nonsense; they’re things that just annoy the physics, and annoy your mind while you’re trying to analyze something. You should be able to do calculations as quickly as possible, and with a minimum of errors. That requires nothing but rote practice—that’s the only way to do it. It’s like making yourself a multiplication table, like you did in elementary school: they’d put a bunch of numbers on the board, and you’d go: “This times that, this times that,” and so on—Bing! Bing! Bing!

1-4 Differentiation

In the same way you must learn differentiation. Make a card, and on the card write a number of expressions of the following general type: for example,

$$\begin{array}{l} 1 + 6t \\ 4t^2 + 2t^3 \\ (1 + 2t)^3 \\ \sqrt{1 + 5t} \\ (t + 7t^2)^{1/3} \end{array} \tag{1.1}$$

and so on. Write, say, a dozen of these expressions. Then, every once in a while, just take the card out of your pocket, put your finger on an expression, and read out the derivative.

In other words, you should be able to see right away:

$$\frac{d}{dt}(1+6t) = 6 \textit{ Bing!}$$
$$\frac{d}{dt}(4t^2+2t^3) = 8t+6t^2 \textit{ Bing!} \tag{1.2}$$
$$\frac{d}{dt}(1+2t)^3 = 6(1+2t)^2 \textit{ Bing!}$$

See? So the first thing to do is to memorize how to do derivatives—cold. That's a necessary practice.

Now, for differentiating more complicated expressions, the derivative of a sum is easy: it's simply the sum of the derivatives of each separate summand. It isn't necessary at this stage in our physics course to know how to differentiate expressions any more complicated than those above, or sums of them, so that in the spirit of this review, I shouldn't tell you any more. But there is a formula for differentiating complicated expressions, which is usually not given in calculus class in the form that I'm going to give it to you, and it turns out to be very useful. You won't learn it later, because nobody will ever tell it to you, but it's a good thing to know how to do.

Suppose I want to differentiate the following:

$$\frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{3/2}} + \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}}. \tag{1.3}$$

Now, the question is how to do it with *dispatch.* Here's how you do it with dispatch. (These are just rules; it's the level to which I've reduced the mathematics, because we're working with the guys who can barely hold on.) Watch!

You write the expression down again, and after each summand you put a bracket:

$$\begin{aligned} &\frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{3/2}} \cdot \Bigg[\\ &+ \frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \cdot \Bigg[\end{aligned} \tag{1.4}$$

Next, you're going to write something inside the brackets, such that when you're all finished, you'll have the derivative of the original expression. (That's why you write the expression down again, in case you don't want to lose it.)

Now, you look at each term and you draw a bar—a divider—and you put the term in the denominator: The first term is $1 + 2t^2$; that goes in the denominator. The power of the term goes in front (it's the first power, 1), and the derivative of the term (by our practice game), $4t$, goes in the numerator. That's one term:

$$\frac{6(1 + 2t^2)(t^3 - t)^2}{\sqrt{t + 5t^2}(4t)^{3/2}} \cdot \left[1\frac{4t}{1 + 2t^2}\right. \\ + \frac{\sqrt{1 + 2t}}{t + \sqrt{1 + t^2}} \cdot \left[\vphantom{\frac{1}{1}}\right. \tag{1.5}$$

(What about the 6? Forget it! Any number in front doesn't make any difference: if you wanted to, you could start out, "6 goes in the denominator; its power, 1, goes in front; and its derivative, 0, goes in the numerator.")

Next term: $t^3 - t$ goes in the denominator; the power, $+2$, goes in front; the derivative, $3t^2 - 1$, goes in the numerator. The next term, $t + 5t^2$, goes in the denominator; the power, $-1/2$ (the inverse square root is a *negative* half power), goes in front; the derivative, $1 + 10t$, goes in the numerator. The next term, $4t$, goes in the denominator; its power, $-3/2$, goes in front; its derivative, 4, goes in the numerator. Close the bracket. That's one summand:

$$\frac{6(1 + 2t^2)(t^3 - t)^2}{\sqrt{t + 5t^2}(4t)^{3/2}} \cdot \left[1\frac{4t}{1 + 2t^2} + 2\frac{3t^2-1}{t^3 - t} - \frac{1}{2}\frac{1 + 10t}{t + 5t^2} - \frac{3}{2}\frac{4}{4t}\right] \\ + \frac{\sqrt{1 + 2t}}{t + \sqrt{1 + t^2}} \cdot \left[\vphantom{\frac{1}{1}}\right. \tag{1.6}$$

Next summand, first term: the power is $+1/2$. The object whose power we're taking is $1 + 2t$; the derivative is 2. The power of the next term, $t + \sqrt{1 + t^2}$, is -1. (You see, it's a reciprocal.) The term goes in the denominator, and its derivative (this is the only hard one, relatively) has two pieces, because it's a sum: $1 + \frac{1}{2}\frac{2t}{\sqrt{1 + t^2}}$. Close the bracket:

$$\frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}(4t)^{3/2}} \cdot \left[1\frac{4t}{1+2t^2} + 2\frac{3t^2\text{–}1}{t^3-t} - \frac{1}{2}\frac{1+10t}{t+5t^2} - \frac{3}{2}\frac{4}{4t}\right]$$
$$+\frac{\sqrt{1+2t}}{t+\sqrt{1+t^2}} \cdot \left[\frac{1}{2}\frac{2}{(1+2t)} - 1\frac{1+\frac{1}{2}\frac{2t}{\sqrt{1+t^2}}}{t+\sqrt{1+t^2}}\right]. \tag{1.7}$$

That's the derivative of the original expression. So, you see, that by memorizing this technique, you can differentiate *anything*—except sines, cosines, logs, and so on, but you can learn the rules for those easily; they're very simple. And then you can use this technique even when the terms include tangents and everything else.

I noticed when I wrote it down you were worried that it was such a complicated expression, but I think you can appreciate now that this is a really powerful method of differentiation because it gives the answer—*boom*—without any delay, no matter how complicated.

The idea here is that the derivative of a function $f = k \cdot u^a \cdot v^b \cdot w^c \ldots$ with respect to t is

$$\frac{df}{dt} = f \cdot \left[a\frac{du/dt}{\mathrm{u}} + b\frac{dv/dt}{\mathrm{v}} + c\frac{dw/dt}{\mathrm{w}} + \cdots\right] \tag{1.8}$$

(where k and a, b, c ... are constants).

However, in this physics course, I doubt any of the problems will be that complicated, so we probably won't have any opportunity to use this. Anyway, that's the way *I* differentiate, and I'm pretty good at it now, so there we are.

1-5 Integration

Now, the opposite process is integration. You should equally well learn to integrate as rapidly as possible. Integration is not as easy as differentiation, but you should be able to integrate simple expressions in your head. It isn't necessary to be able to integrate every expression; for example, $(t+7t^2)^{1/3}$ is not possible to integrate in an easy fashion, but the others below are. So, when you choose expressions to practice integration, be careful that they can be done easily:

$$\int (1+6t)\,dt = t + 3t^2$$

$$\int (4t^2+2t^3)\,dt = \frac{4t^3}{3} + \frac{t^4}{2}$$